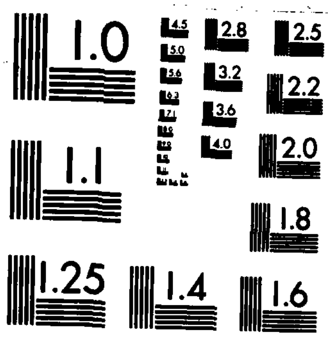
AD-A155 096 BISTABILITY BASINS OF ATTRACTION AND PREDICTABILITY IN A FORCED MASS-REACTION MODEL(U) NAVAL RESEARCH LAB WASHINGTON DC I B SCHWARTZ 13 MAY 85 NRL-MR-5538 1/1

UNCLASSIFIED F/G 12/1 NL

END
FILMED
DTIC


1.0
1.1
1.25
1.4
1.6
1.8
2.0
2.2
2.5
2.8
3.2
3.6
4.0
4.5
5.0
5.6
6.3

NRL Memorandum Report 5538

Bistability, Basins of Attraction, and Predictability in a Forced Mass-Reaction Model

I. B. SCHWARTZ

Electro-optical Technology Branch
Optical Sciences Division

May 13, 1985

NAVAL RESEARCH LABORATORY
Washington, D.C.

SECURITY CLASSIFICATION OF THIS PAGE

REPORT DOCUMENTATION PAGE

1a. REPORT SECURITY CLASSIFICATION	UNCLASSIFIED
1b. RESTRICTIVE MARKINGS	
2a. SECURITY CLASSIFICATION AUTHORITY	
2b. DECLASSIFICATION / DOWNGRADING SCHEDULE	
3. DISTRIBUTION / AVAILABILITY OF REPORT	Approved for public release; distribution unlimited.
4. PERFORMING ORGANIZATION REPORT NUMBER(S)	NRL Memorandum Report 5538
5. MONITORING ORGANIZATION REPORT NUMBER(S)	
6a. NAME OF PERFORMING ORGANIZATION	Naval Research Laboratory
6b. OFFICE SYMBOL (If applicable)	6520
7a. NAME OF MONITORING ORGANIZATION	Office of Naval Research
6c. ADDRESS (City, State, and ZIP Code)	Washington, DC 20375-5000
7b. ADDRESS (City, State, and ZIP Code)	Arlington, VA 22217
8a. NAME OF FUNDING / SPONSORING ORGANIZATION	Office of Naval Research
8b. OFFICE SYMBOL (If applicable)	
9. PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER	
8c. ADDRESS (City, State, and ZIP Code)	Arlington, VA 22217

10. SOURCE OF FUNDING NUMBERS

PROGRAM ELEMENT NO.	PROJECT NO.	TASK NO.	WORK UNIT ACCESSION NO.
	ONR 611S3N		DN780-122

11. TITLE (Include Security Classification)
Bistability, Basins of Attraction, and Predictability in a Forced Mass-Reaction Model

12. PERSONAL AUTHOR(S)
Schwartz, I.B.

13a. TYPE OF REPORT	13b. TIME COVERED	14. DATE OF REPORT (Year, Month, Day)	15. PAGE COUNT
Memorandum	FROM 9-15-83 TO 9-15-84	1985 May 13	15

16. SUPPLEMENTARY NOTATION

17. COSATI CODES

FIELD	GROUP	SUB-GROUP

18. SUBJECT TERMS (Continue on reverse if necessary and identify by block number)
Bistability; Basin boundaries; Predictability; Dimension; Mass-reaction; Subharmonic orbits

19. ABSTRACT (Continue on reverse if necessary and identify by block number)

A simple mass-reaction model with periodically forced contact rate is considered. It is shown that the forced differential equation exhibits at least two distinct periodic orbits for several ranges of forcing amplitude. Basins of attraction are computed for co-existing stable periodic orbits and are shown to be intertwined in a complex manner. Dimension (capacity) of the basin is computed, and it is shown how the basin structure affects final state predictability.

20. DISTRIBUTION / AVAILABILITY OF ABSTRACT	☒ UNCLASSIFIED/UNLIMITED ☐ SAME AS RPT ☐ DTIC USERS
21. ABSTRACT SECURITY CLASSIFICATION	UNCLASSIFIED
22a. NAME OF RESPONSIBLE INDIVIDUAL	I.B. Schwartz
22b. TELEPHONE (Include Area Code)	202-767-3799
22c. OFFICE SYMBOL	6520

DD FORM 1473, 84 MAR — 83 APR edition may be used until exhausted. All other editions are obsolete.

SECURITY CLASSIFICATION OF THIS PAGE

CONTENTS

BISTABILITY, BASINS OF ATTRACTION, AND PREDICTABILITY IN A FORCED MASS-REACTION MODEL

1. INTRODUCTION

Bistable phenomena can occur in many physical, chemical, and biological models of natural phenomena [e.g., 1, 2, 3, 4, 5, 6, 7]. An important subset of problems exhibiting bistability consists of those models employing mass-reaction kinetics. Crucial to understanding any mass-reaction model is a knowledge of the parameter corresponding to the rate of contact between two or more species. In certain applications, the contact rate may be time dependent, and in fact, periodic. For example, the process of temporally increasing and decreasing the solar intensity respectively changes the probability of contact between two reacting species in the atmosphere [8, 9, 10]. In addition to perturbing reactants in the atmosphere, periodic forcing of contact rates plays an important role in modelling recurrent epidemic outbreaks [11, 7]. It is important to note that both physical and biological phenomena exhibit oscillations which are longer than the forcing period, or not periodic at all [10, 12]. Furthermore, it is not uncommon for periodically forced differential equation models to exhibit two or more stable subharmonic solutions for a given set of parameters [e.g., 2, 7, 13]. The question we consider here is, how well can one predict the asymptotic final state given initial conditions having finite precision for a problem that exhibits two different stable periodic orbits.

In particular, ~~we~~ this document considers a simple mass-reaction model with a periodically forced contact rate. The following is shown numerically:

~~(i)~~ 1) There exist parameter values for which the model exhibits at least two distinct stable subharmonic periodic orbits.

~~(ii)~~ 2) The basins of attraction of each orbit can be very complicated, thus affecting final state predictability as a function of precision in initial conditions.

~~(iii)~~ 3) The dimension (capacity) of the basin boundary is estimated using the techniques in [14].

2. A SIMPLE MASS-REACTION MODEL

Let c_i denote the fraction of concentrations, $i = 1, 4$. The model used is:

$$\begin{aligned} c_1' &= \mu(1 - c_1) - \beta(t)c_1c_3 \\ c_2' &= \beta(t)c_1c_3 - (\alpha + \mu)c_2 \qquad \text{(MR)} \\ c_3' &= \alpha c_2 - (\gamma + \mu)c_3 \\ c_4' &= \gamma c_3 - \mu c_4, \end{aligned}$$

Manuscript approved January 2, 1985.

where μ, α, γ are constants. Periodicity in the contact rate is incorporated by assuming $\beta(t) = \beta_0(1 + \beta_1 \cos 2\pi t)$, and the forcing amplitude, β_1, is the parameter to be varied. Note that $\sum_{i=1}^{4} c_i = 1$ for all time, and therefore it is sufficient to study the first three equations in (MR), c_4 being given by $1 - \sum_{i=1}^{3} c_i$.

When $\beta_1 = 0$, equation (MR) has two steady states: $(1, 0, 0, 0)$ and $\mathcal{L}^0 = (c_1^0, c_2^0, c_3^0, c_4^0)$ where $c_1^0 = (\mu + \alpha)(\mu + \gamma)/\beta_0\alpha = 1/Q$, $c_2^0 = (\mu + \gamma)\mu(Q - 1)/\beta_0\alpha$, $c_3^0 = \mu(Q - 1)/\beta_0$. Parameters are chosen so that $(1, 0, 0, 0)$ is unstable and $\mathcal{L}^0$ is stable. Assume $Q > 1$, $\mu(Q - 1)$ is small, and introduce the change of variables: $\mu(Q - 1) = \epsilon$, $\mu + \alpha = \Delta_2/\epsilon$, $\mu + \gamma = \Delta_3/\epsilon$, where ϵ is a small parameter and Δ_2, Δ_3 are positive constants. Letting $c_i = c_i^0(1 + x_i)$, $i = 1, 2, 3$ and using the above scaling results in the system:

$$
\begin{aligned}
x_1' &= -\epsilon[(\eta + \beta_1 \cos 2\pi t)x_1 + (1 + \beta_1 \cos 2\pi t)x_3 \\
&\qquad + \beta_1 \cos 2\pi t + x_1x_3(1 + \beta_1 \cos 2\pi t)] \\
x_2' &= \Delta 2[\beta_1 \cos 2\pi t + (x_1 + x_3)(1 + \beta_1 \cos 2\pi t) \qquad\qquad \text{(MS)} \\
&\qquad - x_2 + x_1x_3(1 + \beta_1 \cos 2\pi t)/\epsilon \\
x_3' &= \Delta_3(x_2 - x_3)/\epsilon, \text{ where } \eta = Q/(Q - 1) > 1.
\end{aligned}
$$

The non-trivial equilibrium point is moved to the origin, and the eigenvalues of the linearized vector field at the origin are given by $\lambda \pm (\epsilon) = \epsilon r \pm i\nu + 0(\epsilon^2)$, $\lambda_3(\epsilon) = -(\Delta_2 + \Delta_3)/\epsilon + 0(\epsilon)$ where $r = [\Delta_2\Delta_3 - (\Delta_2 + \Delta_3)^2\eta]/2(\Delta_2 + \Delta_3)^2 < 0$, and $\nu^2 = \Delta_2\Delta_3/(\Delta_2 + \Delta_3)$. Thus there is a strong attraction onto a surface in the direction corresponding to λ_3, and a weak attraction in which orbits slowly spiral into the origin. Since the orbits appear to lie approximately in two dimensions where $c_2(t) \approx (\mu + \gamma)c_3(t)/\alpha$, we examine variations in c_1 and c_3. The initial conditions (c_1, c_2, c_3), are restricted to satisfying $c_2 = c_3(\mu + \gamma)/\alpha$ from here on.

3. BISTABLE BEHAVIOR

For given values of $\beta_1 > 0$, there exist periodic orbits which have period 1 as well as subharmonic periods. Both stable and unstable orbits were computed as a function of β_1 using the techniques in [15]. The fixed values of the parameters used in the computations are listed in Table 1. If $\Phi(t)$ denotes the vector solution to (MS) of a periodic solution of period n for a given value of β_1, define the norm of Φ as

$$||\Phi(t)||^2 = (1/n)\int_0^n \Phi(t)\cdot\Phi(t)\,dt.$$

A plot of $||\Phi||$ as a function of β_1 is shown in Fig. 1. Emanating from the stable steady state is a small amplitude period 1 orbit. At a critical value of β_1, the period 1 orbit goes unstable (dashed line) and a stable period 2 bifurcates from the period 1 branch. Notice that there also exist large amplitude saddle-node (stable-unstable) bifurcations of periods 3 and 4. Examples of the period 3 and period 2 orbits are shown in Figs. 2a and 2b. From Fig. 1, it is clear that there exist at least two stable periodic orbits for certain values of β_1. Thus the model exhibits bistability for several ranges of forcing amplitude.

Table 1 — Parameter Values Used for Numerical Simulation

β_0	=	1575
μ	=	0.02
α	=	1/0.0279
λ	=	1/0.01

4. PREDICTABILITY AND DIMENSION OF THE BASINS OF ATTRACTION

When $\beta_1 = 0.1$, Fig. 1 shows that there co-exist stable period 1 (SP1) and stable period 3 (SP3) orbits. To find the basins of attraction, we simplify the problem by assuming initial conditions, (c_1, c_2, c_3), are restricted by satisfying $c_2 = ((\mu + \gamma)/\alpha)c_3$ as described above. We pick a pair (c_1, c_3) at random using a uniform probability distribution and then determine if the trajectory converges to an SP1 or SP3 orbit. If it converges to SP1, (c_1, c_3) is plotted. Otherwise, the trajectory converges to SP3 and the point is not plotted. (No other periodic orbit was observed.) Figure 3a illustrated the basin of attraction for SP1 (dotted) over a given region. The blank areas denote the basin for SP3. A uniform grid was initially chosen, and approximately 24 per cent of the initial conditions converged to SP3. The same procedure was performed for a small region in Fig. 3a, except that the points were chosen at random. Figure 3b shows the result. Clearly the basins of attraction for SP1 and SP3 are intertwined in a complex manner.

In [14] it is shown that a discrete map in two dimensions possessing a complicated basin structure results in an obstruction to predictability. Here we follow their procedure for measuring the predictability of equation (MR). Given a random initial condition, (c_1, c_3), we consider perturbed initial conditions $(c_1, c_3 \pm \tilde{\epsilon})$ for a given $\tilde{\epsilon} > 0$. If at least one of the perturbed initial conditions converges to an orbit other than the orbit passing through the unperturbed initial condition, we say (c_1, c_3) is uncertain. Four thousand random initial points were chosen, and the fraction, F, of uncertain initial points was computed. The procedure was then repeated for several values of $\tilde{\epsilon}$.

Figure 4 illustrates the scaling of F as a function of $\tilde{\epsilon}$. Using a linear least squares fit, it is found that F is proportional to $\tilde{\epsilon}^{0.68}$. Thus the capacity [16] of the basin boundary is approximately 1.32.

5. CONCLUSIONS

We have demonstrated bistability for a simple periodically forced mass-reaction model. Furthermore, we have illustrated that the basins of attraction of the periodic orbits are intertwined. The effect of the complicated basin structure is to obstruct the predictability of the final state given imprecision in initial conditions. That is, given the slope of 0.68 in Fig. 4, to decrease the fraction of uncertain initial conditions in phase space by an order of magnitude requires more than an order of magnitude reduction in uncertainty. Therefore, requiring a high degree of predictability can result in the need to perform extraordinary high precision computations.

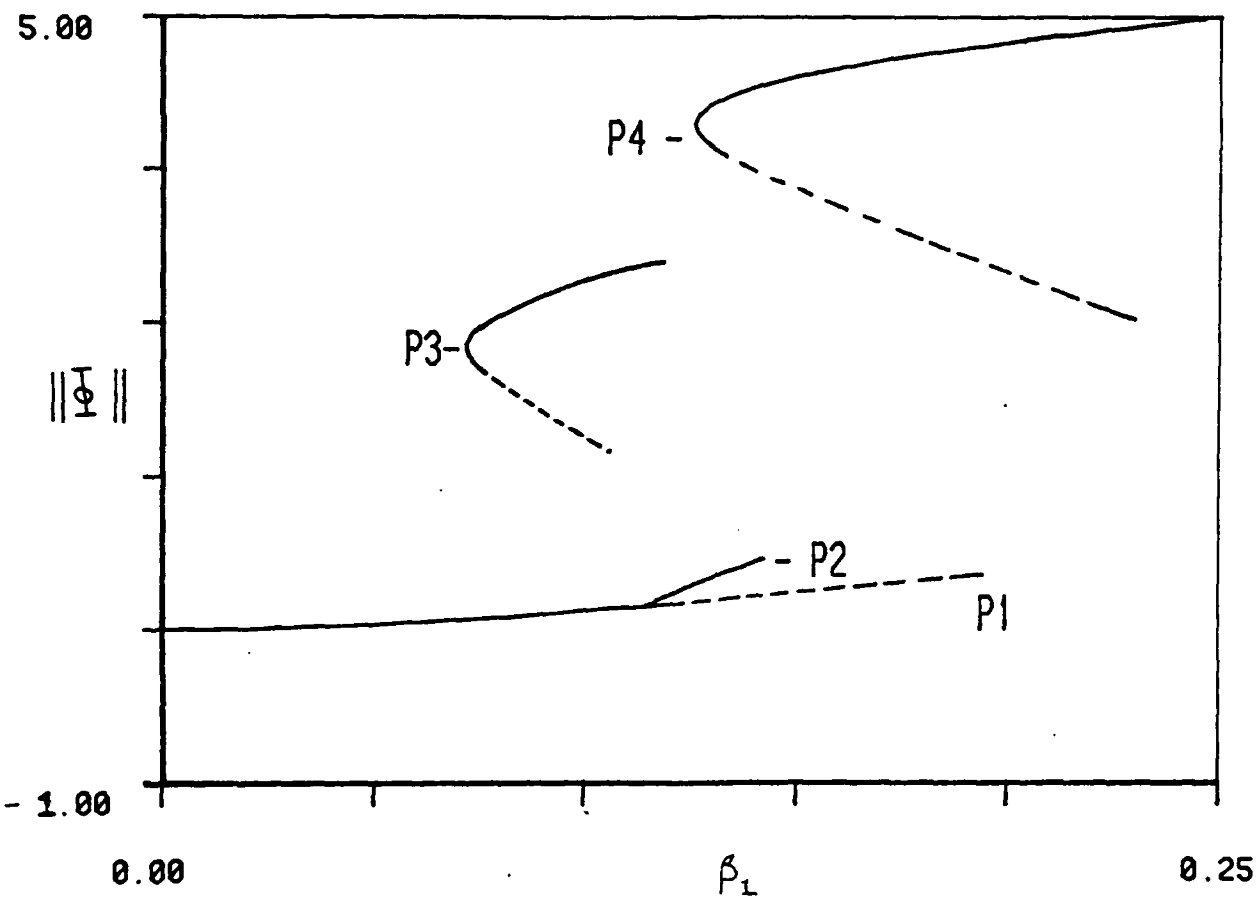


Fig. 1 — Bifurcation diagram. A plot of the norm of periodic solutions as a function of forcing amplitude shows the existence of bistabiliy. Solid lines denote stable orbits, and dashed lines unstable orbits.

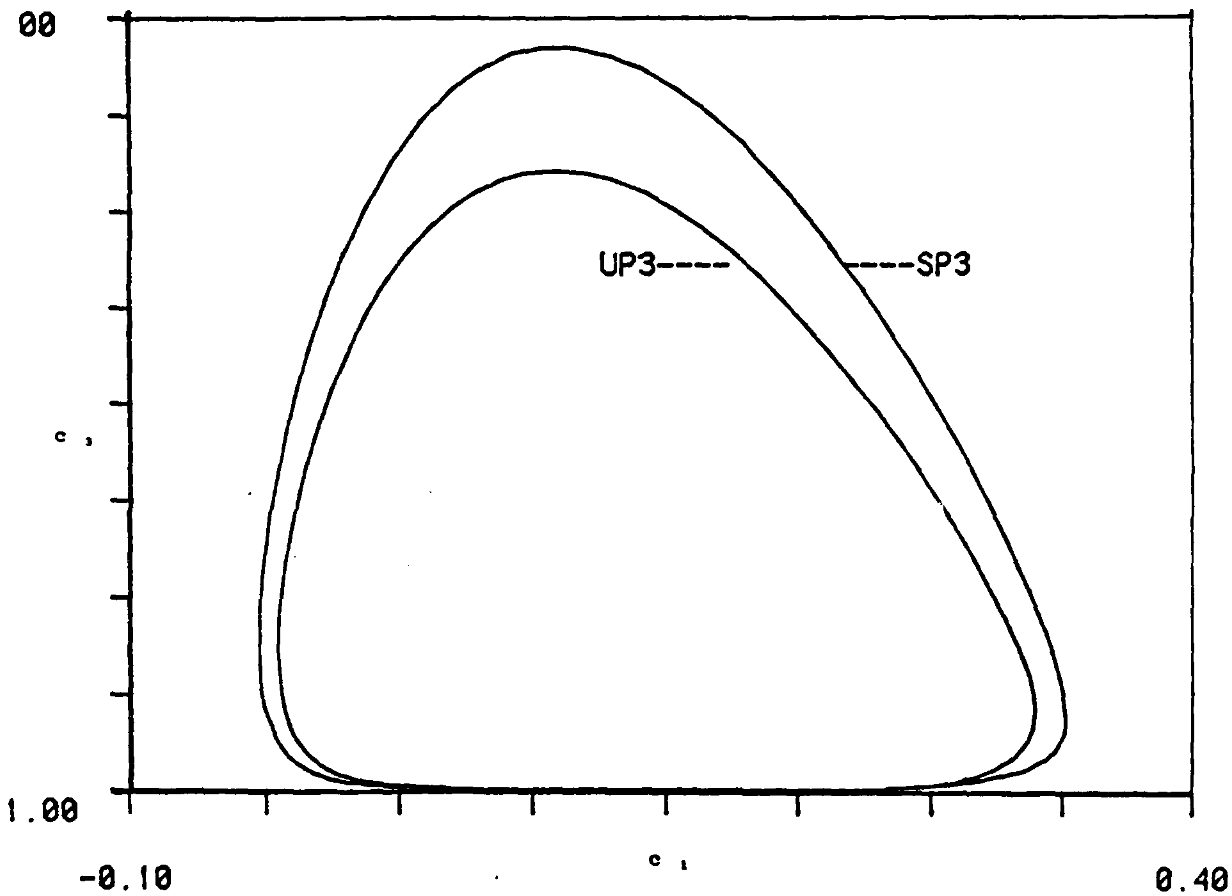


Fig. 2a — An example of unstable and stable period 3 orbits when $\beta_1 \approx 0.075$.

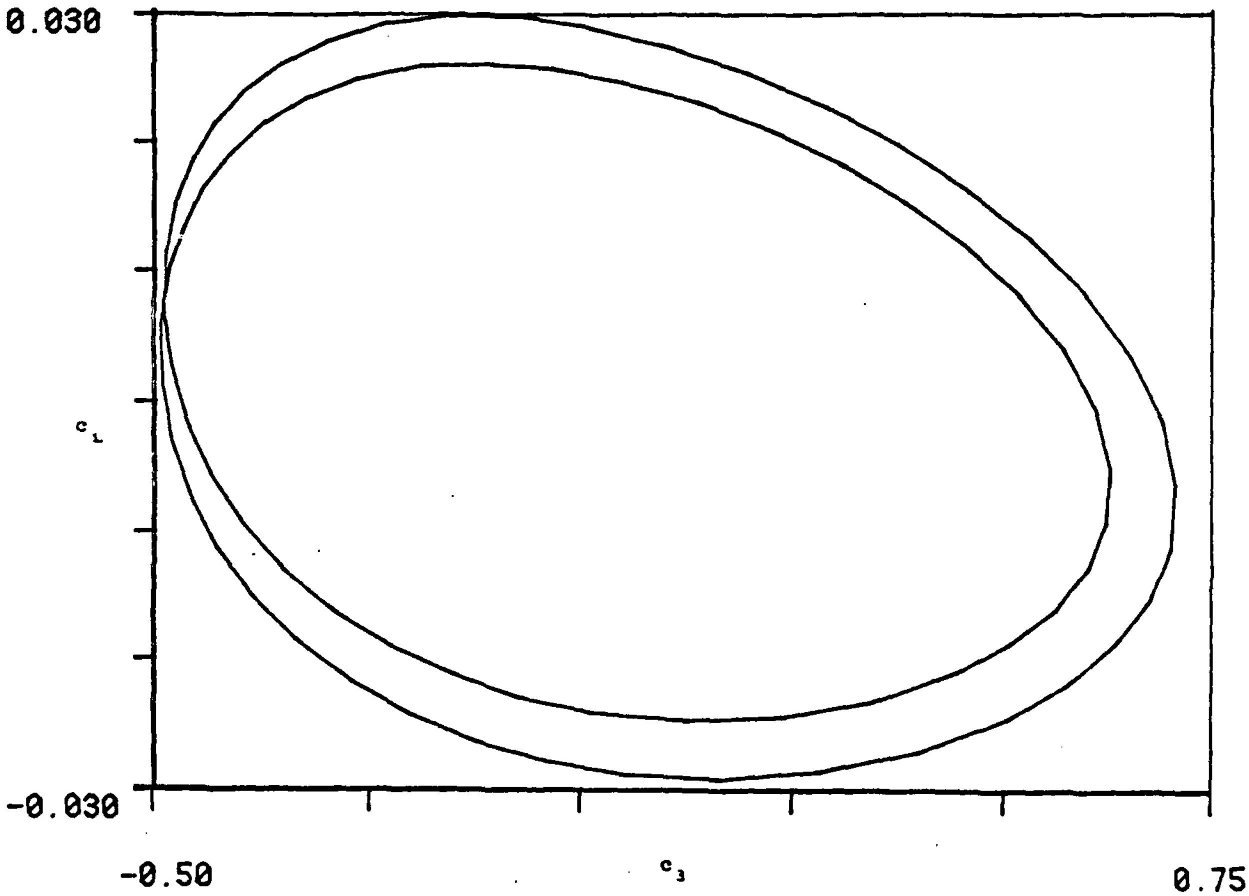


Fig. 2b — A period doubled orbit. Here $\beta_1 \approx 0.12$.

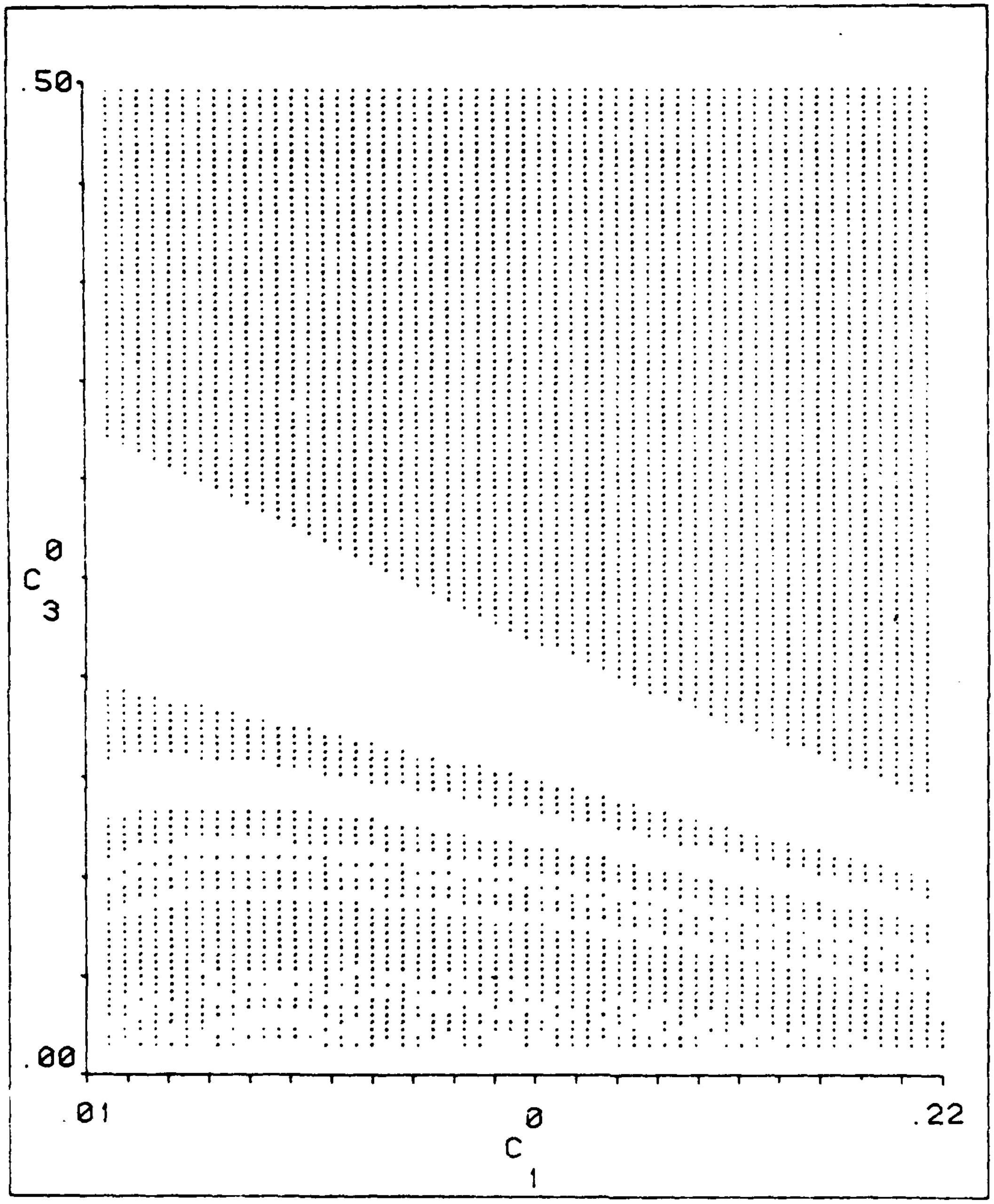


Fig. 3a — Basins of attraction for a bistable case are shown when $\beta_1 = 0.1$. Dotted regions depict the basin of attraction for the SP1 orbit, and the white regions depict the basin for the SP3 cycle. The grid of initial conditions was uniform.

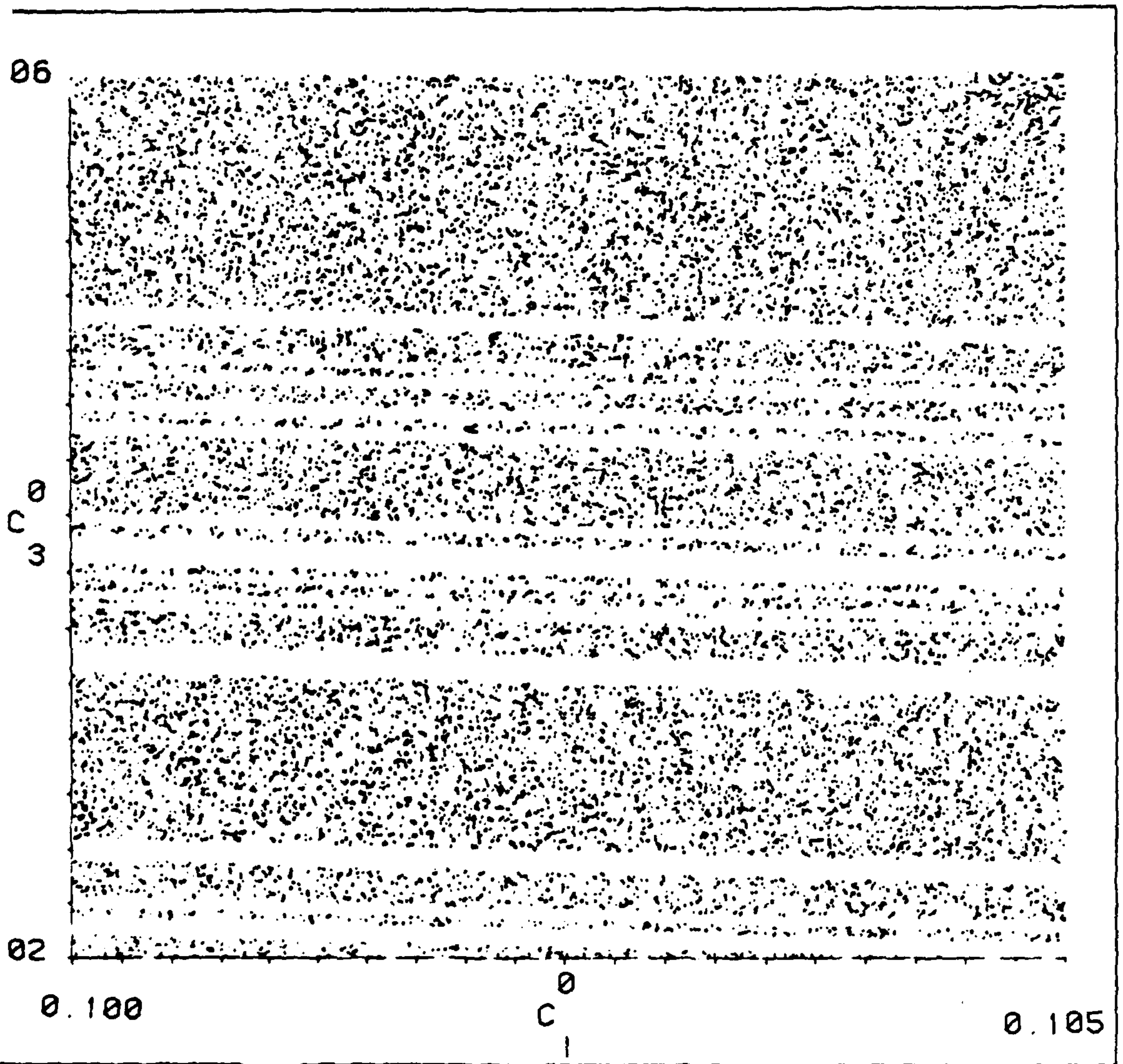


Fig. 3b — Basins of ;action depicting a blow-up of a small region on Fig. 3a. The initial conditions ere chosen at random. The fractal-like structure is evident.

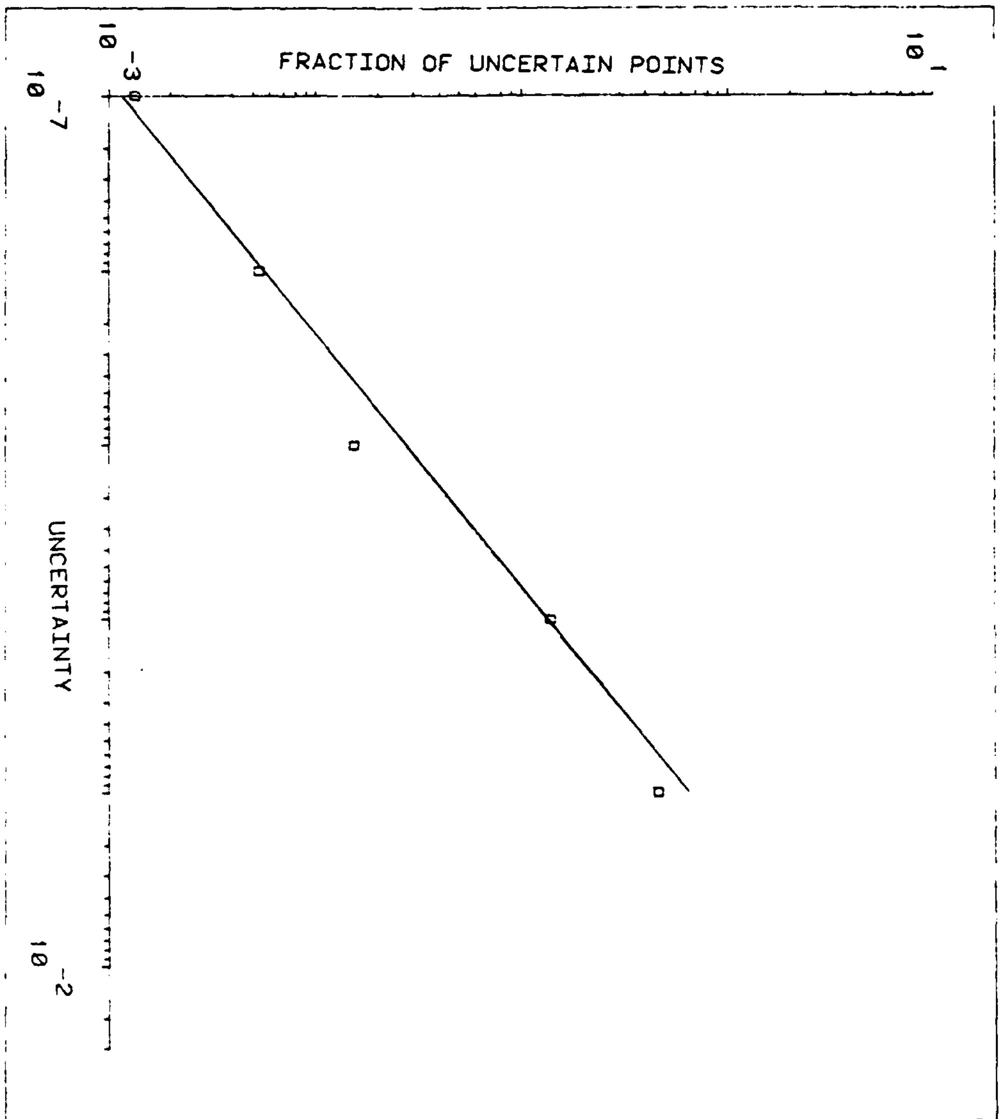


Fig. 4 — Fraction of uncertain points are plotted as a function of imprecision in the initial conditions. The dimension of the basin boundary was found to be approximately 1.32.

REFERENCES

[1] E. N. Lorenz, J. Atmos. Sci. **20** (1963) 130.

[2] L. Flatto and N. Levinson, J. Rat. Mech. Anal. **4** (1955) 943.

[3] F. A. Hopf, D. L. Kaplan, H. M. Gibbs, and R. L. Shoemaker, Phys. Rev. A **25** (1982) 2172.

[4] K. Ikeda and H. Daido, Phys. Rev. Letters **45** (1980) 709.

[5] M. Levy, *Qualitative Analysis of Periodically Forced Relaxation Oscillations* American Mathematical Society (1980).

[6] J. Rinzel and I. B. Schwartz, J. Chem. Phys. **80** (1984) 5610.

[7] I. B. Schwartz and H. L. Smith, J. Math. Biol. **18** (1983) 233.

[8] W. Chameides and J. C. G. Walker, J. Geophys. Res. **78** (1973) 8751.

[9] R. G. Fleagle and J. A. Businger, *An Introduction to Atmospheric Physics* Academic Press (1963). Refer to pages 167-169.

[10] A. Q. Eschenroeder, Atmospheric concentrations of photo-chemical oxidants and models for predicting air quality, in *Ozone and Other Photo-chemical Oxidants*, National Research Council, National Academy of Sciences (1977).

[11] W. P. London and J. A. Yorke, Amer. J. Epidemiology **98** (1973) 453.

[12] H. Emerson, Amer. J. Pub. Health **27** (1937) 1.

[13] S. N. Chow, J. K. Hale, and J. Mallet-Paret, J. Diff. Eqs. **37** (1980) 351.

[14] C. Grebogi, S. McDonald, E. Ott, and J. A. Yorke, Phys. Letters A **99A** (1983) 415.

[15] I. B. Schwartz, SIAM J. Numer. Anal. **20** (1983) 106.

[16] J. D. Farmer, E. Ott, J. A. Yorke, Physica **7D** (1983) 153.

END

FILMED

7-85

DTIC